

# Approximate Integration

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Sometimes, it is impossible to find the exact value of a definite integral.  
For example,

$$\int_0^1 e^{x^2} dx; \quad \int_0^\pi \sin x^2 dx, \quad \int_0^1 \sqrt[4]{1+x^5} dx \dots$$

We need to find approximate values of definite integrals.

Sometimes, **it is impossible** to find the **exact value** of a definite integral.  
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We need to find **approximate values** of definite integrals.

We already known one method for approximate integration: **any Riemann sum could be used as an approximation to the integral.**

- If we divide  $[a, b]$  into  $n$  subintervals of equal length  $\Delta x = (b - a)/n$ , we have:

$$\int_a^b f(x) dx \approx \sum_{i=1}^n f(c_i) \Delta x$$

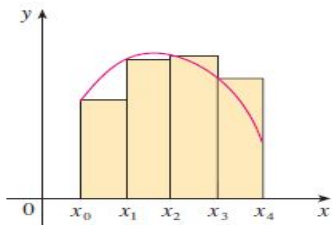
where  $c_i$  is any point in the  $i$ th subinterval  $[x_{i-1}, x_i]$ .

# Left endpoint approximation

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- If  $c_i$  is chosen to be the **left endpoint of the interval**, then  $c_i = x_{i-1}$  and we have the **left endpoint approximation**:

$$\int_a^b f(x) dx \approx L_n = \sum_{i=1}^n f(x_{i-1}) \Delta x$$



(a) Left endpoint approximation

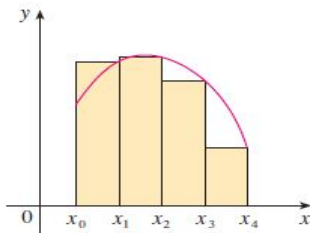
$$L_4 = (f(x_0) + f(x_1) + f(x_2) + f(x_3)) \Delta x$$

# Right endpoint approximation

Cr

- If we choose  $c_i$  to be the right endpoint,  $c_i = x_i$ , then we have the right endpoint approximation:

$$\int_a^b f(x) dx \approx R_n = \sum_{i=1}^n f(x_i) \Delta x$$



(b) Right endpoint approximation

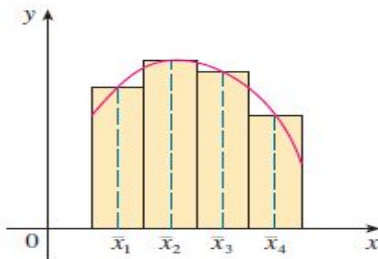
$$R_4 = (f(x_1) + f(x_2) + f(x_3) + f(x_4)) \Delta x$$

# Midpoint approximation

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If we choose  $c_i$  to be the **midpoint point**,  $c_i = \bar{x}_i = \frac{1}{2}(x_{i-1} + x_i)$ , then we have the **midpoint approximation**:

$$\int_a^b f(x) dx \approx M_n = \sum_{i=1}^n f(\bar{x}_i) \Delta x$$



(c) Midpoint approximation

$$M_4 = (f(\bar{x}_1) + f(\bar{x}_2) + f(\bar{x}_3) + f(\bar{x}_4)) \Delta x$$

# Trapezoidal Rule



Another approximation-called the **Trapezoidal Rule** results from averaging the left-and right-endpoint approximations:

$$\begin{aligned}\int_a^b f(x) dx &\approx \frac{1}{2} \left[ \sum_{i=1}^n f(x_{i-1}) \Delta x + \sum_{i=1}^n f(x_i) \Delta x \right] = \frac{\Delta x}{2} \left[ \sum_{i=1}^n (f(x_{i-1}) + f(x_i)) \right] \\ &= \frac{\Delta x}{2} [(f(x_0) + f(x_1)) + (f(x_1) + f(x_2)) + \cdots + (f(x_{n-1}) + f(x_n))] \\ &= \frac{\Delta x}{2} [f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]\end{aligned}$$

## Trapezoidal Rule

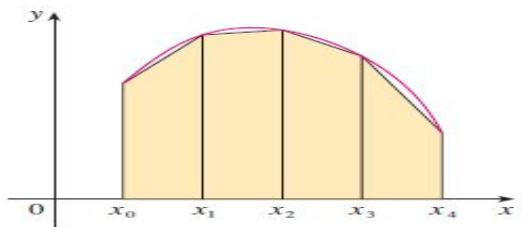
$$\int_a^b f(x) dx \approx T_n = \frac{\Delta x}{2} [f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

where  $\Delta x = (b - a)/n$  and  $x_i = a + i \Delta x$ .

# Trapezoidal Rule

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$$T_n = \frac{R_n + L_n}{2} = \text{the sum of the areas of all the trapezoids}$$



**FIGURE 2**  
Trapezoidal approximation



Example: Use the Trapezoidal Rule with  $n = 5$  to approximate the integral

$$\int_1^2 \frac{1}{x} dx.$$

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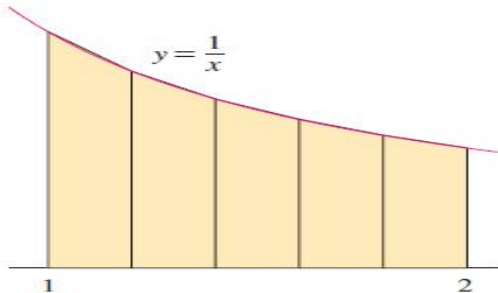
**Solution:** With  $n = 5$ ,  $a = 1$  and  $b = 2$ , we have  $\Delta x = \frac{2-1}{5} = 0.2$ , and so the Trapezoidal Rule gives

$$\begin{aligned}\int_1^2 \frac{1}{x} dx &\approx T_5 = \frac{0.2}{2} [f(1) + 2f(1.2) + 2f(1.4) + 2f(1.6) + 2f(1.8) + f(2)] \\ &= 0.1 \left[ \frac{1}{1} + \frac{2}{1.2} + \frac{2}{1.4} + \frac{2}{1.6} + \frac{2}{1.8} + \frac{1}{2} \right] \\ &\approx 0.695635\end{aligned}$$

# Trapezoidal Rule

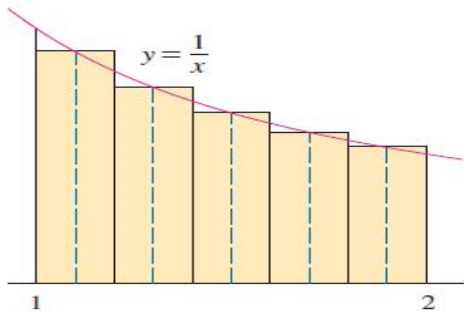
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$$\int_0^1 \frac{1}{x} dx \approx T_5$$



Example: Use the **Midpoint Rule** with  $n = 5$  to approximate the integral

$$\int_1^2 \frac{1}{x} dx.$$



**Solution:** With  $n = 5$ ,  $a = 1$  and  $b = 2$ , we have  $\Delta x = \frac{2-1}{5} = 0.2$ .

The midpoints of the five subintervals are 1.1, 1.3, 1.5, 1.7 and 1.9, so the Midpoint Rule gives

$$\begin{aligned}\int_1^2 \frac{1}{x} dx &\approx \Delta x [f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9)] \\ &= 0.2 \left[ \frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9} \right] \approx 0.691908.\end{aligned}$$

# Error bounds in approximations

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The Midpoint Rule gives  $\int_1^2 \frac{1}{x} dx \approx 0.691908$ .

The Trapezoidal Rule gives  $\int_1^2 \frac{1}{x} dx \approx 0.6956235$ .

By the Fundamental Theorem of Calculus,

$$\int_1^2 \frac{1}{x} dx = \ln x \Big|_1^2 \approx 0.693147.$$

The **error** in using an approximation is defined to be the amount that needs to be added to the approximation to make it exact. So the Trapezoidal and Midpoint Rule approximations for  $n = 5$  are

$$E_T \approx -0.002488 \quad \text{and} \quad E_M \approx 0.001239.$$

In general, we have

$$E_T = \int_a^b f(x) dx - T_n; \quad E_M = \int_a^b f(x) dx - M_n.$$

### Theorem (Error Estimate for the Trapezoidal Rule)

If  $f$  has a continuous second derivative on  $[a, b]$  and satisfies  $|f''(x)| \leq K$  there, then

$$|E_T| \leq \frac{K(b-a)^3}{12n^2}.$$

Let's apply this error estimate to the Trapezoidal Rule approximation in the above example. If  $f(x) = \frac{1}{x}$ , then  $f''(x) = \frac{2}{x^3}$ . Since  $1 \leq x \leq 2$ , we have

$$|f''(x)| = \left| \frac{2}{x^3} \right| \leq 2.$$

Therefore, taking  $K = 2$ ,  $a = 1$  and  $b = 2$  in the above error estimate, we see that

$$|E_T| \leq \frac{2(2-1)^3}{12 \cdot 5^2} \approx 0.006667.$$

### Theorem 6.2 (Error Estimate for the Midpoint Rule)

If  $f$  has a continuous second derivative on  $[a, b]$  and satisfies  $|f''(x)| \leq K$  on  $[a, b]$ , then

$$|E_M| \leq \frac{K(b-a)^3}{24n^2}.$$



# THE SIMPSON'S RULE



As before, we divide  $[a, b]$  into  $n$  subintervals of equal length  $h = \Delta x = (b - a)/n$ . However, this time, we assume  $n$  is an **even** number. Then, on each consecutive pair of intervals, we approximate the curve  $y = f(x) \geq 0$  by a parabola. If  $y_i = f(x_i)$ , then  $P_i(x_i, y_i)$  is the point on the curve lying above  $x_i$ . The area under the parabola through  $P_i$ ,  $P_{i+1}$ , and  $P_{i+2}$  is

$$\frac{\Delta x}{3} [f(x_i) + 4f(x_{i+1}) + f(x_{i+2})].$$

So,

$$\int_{x_i}^{x_{i+2}} f(x) dx \approx \frac{\Delta x}{3} [f(x_i) + 4f(x_{i+1}) + f(x_{i+2})].$$

Adding these  $n/2$  individual approximations we get the Simpson's Rule approximation to the integral  $\int_a^b f(x) dx$ .

**Definition 6.3 (Simpson's Rule)**

Assume that  $n$  is even. Let  $\Delta x = (b - a)/n$  and  $y_j = f(a + j\Delta x)$ . The  $n$ th approximation to  $\int_a^b f(x)dx$  by Simpson's Rule is

$$S_n = \frac{\Delta x}{3} (y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n)$$

**Example 6.4** Calculate the approximations  $S_4$  and  $S_8$  for  $I = \int_1^2 \frac{1}{x} dx$  and compare them with the actual value  $I = \ln 2 = 0.69314718$ , and with the values of  $T_4$ ,  $T_8$ ,  $M_4$  and  $M_8$ .

## Error Bound

### Theorem 6.3 (Error Estimate for Simpson's Rule)

If  $f$  has a continuous fourth derivative on  $[a, b]$  and satisfies  $|f^{(4)}(x)| \leq K$  there, then

$$|E_S| = \left| \int_a^b f(x) dx - S_n \right| \leq \frac{b-a}{180} K h^4 = \frac{K(b-a)^5}{180n^4},$$

where  $h = (b-a)/n$ .

**Example 6.5** The velocity (in miles per hour) of a Piper Cub aircraft traveling due west is recorded every minute during the first 10 min after takeoff. Use the Trapezoid Rule and Simpson's Rule to estimate the distance traveled after 10 min.

|      |   |    |    |    |    |     |    |    |    |    |    |
|------|---|----|----|----|----|-----|----|----|----|----|----|
| t    | 0 | 1  | 2  | 3  | 4  | 5   | 6  | 7  | 8  | 9  | 10 |
| v(t) | 0 | 50 | 60 | 80 | 90 | 100 | 95 | 85 | 80 | 75 | 85 |